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NON-LINEAR OSCILLATIONS OF A HAMILTONIAN SYSTEM WITH 2:1 RESONANCE†

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Non-linear oscillations of an autonomous Hamiltonian system with two degrees of freedom in the neighbourhood of a stable equilibrium are considered. It is assumed that the frequency ratio of the linear oscillations is close to or equal to two, and that the Hamiltonian is sign-definite in the neighbourhood of the equilibrium. A solution is presented to the problem of the orbital stability of periodic motions emanating from the equilibrium position. Conditionally periodic motions of an approximate system are analysed taking into account terms of order up to and including three in the normalized Hamiltonian. The KAM theory is used to consider the problem of maintaining these motions taking into account fourth- and higher-order terms in the series expansion of the Hamiltonian in a sufficiently small neighbourhood of the equilibrium. The results are used to investigate non-linear oscillations of an elastic pendulum. © 2000 Elsevier Science Ltd. All rights reserved.

1. INTRODUCTION

Consider an autonomous Hamiltonian system with two degrees of freedom, assuming that the origin $q_i = 0$, $p_i = 0$ $(i = 1, 2)$ of the phase space is an equilibrium position of the system and that the Hamiltonian is analytic in a certain neighbourhood of the origin

$$H = H_2 + H_3 + \ldots + H_k + \ldots \tag{1.1}$$

where H_k are forms of degree k in q_i and p_i $(i = 1, 2)$. The quadratic form H_2 is assumed to be positive-definite. Then the equilibrium position is stable and the linearized system is a combination of two harmonic oscillators, whose frequencies we will denote by σ_i $(i = 1, 2)$.

We will investigate non-linear oscillations in the neighbourhood of the equilibrium position on the assumption that there is a third-order resonance, when the frequency ratio of the linear oscillations is close to or equal to two. We put $\sigma_1 = (2 + \mu)\sigma_2$, where $0 \leqslant |\mu| \ll 1$.

A good many publications have been devoted to the analysis of non-linear oscillations in Hamiltonian systems in the presence of resonance. Results of approximate studies have been published for third-order resonance [1–7]. The problem of the existence of resonant periodic motions in a small neighbourhood of an equilibrium position has been analysed rigorously [8, 9], and their orbital stability has been investigated in the first approximation [9].

The purpose of this paper is to give a detailed treatment of the general nature of non-linear conditionally periodic motions in a small neighbourhood of an equilibrium position in the case when the frequencies of the linear oscillations are exactly commensurable ($\mu = 0$), as well as a solution of the non-linear problem of the orbital stability of periodic motions emanating from the equilibrium position in the case of exact or approximate commensurability ($0 \leqslant |\mu| \ll 1$).

We will assume that the canonically conjugate variables q_i and p_i $(i = 1, 2)$ have been chosen (via a normalizing canonical transformation) so that the second- and third-degree terms in expansion (1.1) are in normal form. This means (see, e.g. [10]) that series (1.1) may be written as follows:

$$H = \frac{1}{2}\sigma_1(q_1^2 + p_1^2) + \frac{1}{2}\sigma_2(q_2^2 + p_2^2) + \frac{\sqrt{2}}{4}\sigma_2[q_1(p_2^2 - q_2^2) - 2p_1q_2p_2] + \ldots$$

where the dots stand for terms of degree higher than three in q_i and p_i.

If we introduce, instead of the time t, and independent variable $\tau = \sigma_2 t$ and, allowing for the smallness of the relevant neighbourhood of the origin, make a canonical change of variables $q_i, p_i \to x_i, y_i$, where

$$q_i = \varepsilon x_i, \quad p_i = \varepsilon y_i, \quad i = 1, 2, \quad 0 < \varepsilon \ll 1$$

then the Hamiltonian becomes

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$$H=\frac{1}{2}(2+\mu)(x_1^2+y_1^2)+\frac{1}{2}(x_2^2+y_2^2)+\varepsilon\frac{\sqrt{2}}{4}[x_1(y_2^2-x_2^2)-2y_1x_2y_2]+O(\varepsilon^2) \tag{1.2}$$

2. LYAPUNOV SHORT-PERIOD MOTIONS AND THEIR STABILITY

Since the ratio $\sigma_2{:}\sigma_1$ is not an integer, it follows by Lyapunov's holomorphic integral theorem [11] that a family of short-period motions exists emanating from the equilibrium position $q_i = 0, p_i = 0$ $(i = 1, 2)$. This means (see [12, 13]) that, given a system with Hamiltonian (1.2), a canonical transformation $x_i, y_i \to x_i', y_i'$ exists, convergent for sufficiently small ε and differing from the identical transformation by a quantity of order ε^2, under which the Hamiltonian takes a form whose series expansion in powers of x_i', y_i' includes no terms involving the first powers of the variables x_2', y_2', while the series terms not dependent on x_2 and y_2 are functions of the combination $x_1'^2$, $y_1'^2$ only. This transformation does not affect the zero- and first-order terms in ε written out explicitly in (1.2). The solutions of the transformed system corresponding to short-period motions may be written as

$$x_1'=\sqrt{c}\sin\Omega_1(\tau+\tau_0),\quad y_1'=\sqrt{c}\cos\Omega_1(\tau+\tau_0),\quad x_2'=0,\quad y_2'=0 \tag{2.1}$$

where $c > 0$ and τ_0 are arbitrary constants and the frequency Ω_1 is a function of c with $\Omega_1 = (2 + \mu) + O(\varepsilon^2)$.

To solve the problem of orbital stability of a Lyapunov short-period motion, we apply a preliminary canonical transformation $x_1', y_1', x_2', y_2' \to \theta, \zeta, \xi_2, \eta_2$ which reduces the problem of orbital stability of the periodic motion (2.1) to an equivalent problem: the stability of an autonomous Hamiltonian system with two degrees of freedom with respect to part of the variables. This transformation may be expressed as a superposition of two univalent canonical changes of variables: first $x_1', y_1', x_2', y_2' \to \varphi^*, r^*, x_2^*, y_2^*$ where

$$x_1'=\sqrt{2r^*}\sin\varphi^*,\quad y_1'=\sqrt{2r^*}\cos\varphi^*,\quad x_2'=x_2^*,\quad y_2'=y_2^*$$

followed by a change of variable $\varphi^*, r^*, x_2^*, y_2^* \to \theta, \zeta, \xi_2, \eta_2$ where

$$\varphi^*=\theta,\quad r^*=\zeta-\frac{1}{4}(\xi_2^2+\eta_2^2)$$

$$x_2^*=\cos\frac{\theta}{2}\xi_2+\sin\frac{\theta}{2}\eta_2,\quad y_2^*=-\sin\frac{\theta}{2}\xi_2+\cos\frac{\theta}{2}\eta_2$$

The equations of motion in the new variables correspond to a Hamiltonian

$$\Gamma=(2+\mu)\zeta-\frac{1}{4}\mu(\xi_2^2+\eta_2^2)-\varepsilon\xi_2\eta_2\sqrt{\zeta-\frac{1}{4}(\xi_2^2+\eta_2^2)}+O(\varepsilon^2) \tag{2.2}$$

while the periodic motions (2.1) may be written in the form

$$\theta=\Omega_1(\tau+\tau_0),\quad \zeta=c/2,\quad \xi_2=0,\quad \eta_2=0$$

We now introduce perturbations $\alpha_2, \beta_2, \gamma_1$ by the formulae

$$\xi_2=\alpha_2,\quad \eta_2=\beta_2,\quad \zeta=c/2+\gamma_1$$

The problem of the orbital stability of the periodic motion (2.1) is equivalent to the problem of the stability of a system with Hamiltonian (2.2) with respect to the variables $\alpha_2, \beta_2, \gamma_1$.

The Hamiltonian of the perturbed motion may be written as a series

$$\Gamma=\Gamma_2+\Gamma_4+\ldots \tag{2.3}$$

where the dots stand for the terms of order at least six in $\alpha_2, \beta_2, \sqrt{(|\gamma_1|)}$; Γ_2 and Γ_4 are forms of degree two and four, respectively, in these variables, with $\Gamma_2 = \Omega_1\gamma_1 + \Gamma_2^*$, where Γ_2^* and Γ_4 may be written, neglecting quantities of order ε^2 and higher, in the form

$$\Gamma_2^*=-\tfrac{1}{4}\mu(\alpha_2^2+\beta_2^2)-\varepsilon\sqrt{c/2}\,\alpha_2\beta_2$$

$$\Gamma_4=\tfrac{1}{4}\varepsilon(2c)^{-1/2}[\alpha_2\beta_2(\alpha_2^2+\beta_2^2)-4\alpha_2\beta_2\gamma_1]$$

In the linearized system of equations of the perturbed motion, the variables α_2 and β_2 are separated from θ and γ_1. Let $\mu \sim \varepsilon^\delta$, where $\delta \geqslant 1$. The characteristic equation of a system with Hamiltonian Γ_2^* is

$$\lambda^2 + \tfrac{1}{4}(\mu^2 - 2\varepsilon^2 c)(1 + O(\varepsilon)) = 0 \tag{2.4}$$

If ε is sufficiently small and $|\mu| < \varepsilon\sqrt{2c}$, then Eq. (2.4) has a positive real root, and, by Lyapunov's theorem on stability in the first approximation, the periodic motion (2.1) is orbitally stable. But if

$$|\mu| > \varepsilon\sqrt{2c} \tag{2.5}$$

then the roots of Eq. (2.4) $\pm\Omega_2$, where $\Omega_1 = 1/2\sqrt{(\mu^2 - 2\varepsilon^2 c)}$, are pure imaginary; linear analysis is then no longer adequate for rigorous conclusions to be drawn as to stability. One has to use the Arnol'd–Moser theorem [12, 14].

Omitting the fairly easy arguments, we immediately present the normalized Hamiltonian of the perturbed motion

$$\Gamma = \Omega_1\rho_1 - \sigma\Omega_2\rho_2 + c_{20}\rho_1^2 + c_{11}\rho_1\rho_2 + c_{02}\rho_2^2 + O_3 \tag{2.6}$$

where O_3 stands for all the terms of order greater than two in ρ_1 and ρ_2, $\sigma = \operatorname{sign}\mu$ and the coefficients c_{ij} satisfy the following estimates

$$c_{20} = O(\varepsilon^2),\ \ c_{11} = \frac{\sigma\varepsilon^2}{2\Omega_2} + O(\varepsilon^2),\ \ c_{02} = -\frac{3\sqrt{2c}\varepsilon\mu}{4\Omega_2^2} + O(\varepsilon) \tag{2.7}$$

By previous results [12, 14], a sufficient condition for orbital stability is that the Hamiltonian (2.3) be iso-energetically non-degenerate; this condition may be written as

$$\Delta \equiv c_{20}\Omega_2^2 + \sigma c_{11}\Omega_1\Omega_2 + c_{02}\Omega_1^2 \neq 0$$

Using expression (2.7), we obtain

$$\Delta = -\frac{3\sqrt{2c}\Omega_1}{2\Omega_2^2}(\varepsilon\mu + O(\varepsilon^3))$$

If ε is sufficiently small, Δ is non-zero, and it therefore follows from the Arnol'd–Moser theorem that the periodic motions (2.1) are orbitally stable, provided that condition (2.5) for their stability in the linear approximation is satisfied.

3. FAMILIES OF LONG-PERIOD MOTIONS AND THEIR STABILITY

Applying in succession the two univalent canonical changes of variables defined by

$$x_i = \sqrt{2r_i}\sin\varphi_i,\ \ y_i = \sqrt{2r_i}\cos\varphi_i,\ \ i = 1, 2 \tag{3.1}$$

$$Q_1 = \varphi_1 - 2\varphi_2,\ \ Q_2 = \varphi_2,\ \ P_1 = r_1,\ \ P_2 = 2r_1 + r_2 \tag{3.2}$$

we reduce Hamiltonian (1.2) to the form

$$H = \mu P_1 + P_2 + \varepsilon(P_2 - 2P_1)\sqrt{P_1}\sin Q_1 + O(\varepsilon^2) \tag{3.3}$$

where, by (3.1) and (3.3), necessarily $P_2 \geqslant 2P_1$.

Let us consider the approximate system whose Hamiltonian is defined by (3.3), if quantities $O(\varepsilon^2)$ are dropped. In this system the coordinate Q_2 is cyclic and and integral $P_2 = c = \text{const} > 0$ exists. The variation of the variables Q_1 and P_1 is described by canonical equations with Hamiltonian

$$H = \mu P_1 + \varepsilon f(P_1)\sin Q_1,\ \ f = (c - 2P_1)\sqrt{P_1} \tag{3.4}$$

In the approximate system equilibrium positions $Q_1 = Q_1^*$, $P_1 = P_1^*$ exist. A fairly simple analysis shows that (1) if $\mu < -\varepsilon\sqrt{2c}$, one equilibrium position exists, for which $Q_1 = \pm\,\pi/2$, $\mu + \varepsilon f' = 0$, where f' is the value of the derivative at $P_1 = P^*{}_1$; (2) if $|\mu| < \varepsilon\sqrt{2c}$, two equilibrium positions exist, for which

$Q_1 = \pm\pi/2$, $\mu + f' = 0$; 3) if $\mu > \varepsilon\sqrt{(2c)}$, there will be one equilibrium position, for which $Q_1 = -\pi/2$, $\mu - \varepsilon f' = 0$.

Now, proceeding as previously in [15, 16], one can apply iso-energetic reduction to transform to Whittaker's equations and, using Poincaré's method, show that in the full system with Hamiltonian (3.3) families of long-period solutions emanate from the above equilibrium positions. The period of these motions as functions of τ tends to 2π as $\varepsilon \to 0$.

Noting condition (2.5), one arrives at a known conclusion (see [9]): if a family of Lyapunov motions is stable, a family of long-period motions also exists; as one crosses the boundary $|\mu| < \varepsilon\sqrt{(2c)}$ of the stability domain of the Lyapunov motions into their unstable domain $|\mu| < \varepsilon\sqrt{(2c)}$, yet another family of long-period motions appears. It has been shown [9] that the families of long period motions are orbitally stable in the first approximation.

We will show that these families are also orbitally stable in the rigorous non-linear formulation of the problem. To do so, as in Section 2, we derive the normal form of the Hamiltonian of the perturbed motion and then use the Arnol'd–Moser theorem. The normal form will be analogous to (2.6)

$$H = \delta_1\rho_1 + \delta_2\rho_2 + c_{20}\rho_1^2 + c_{11}\rho_1\rho_2 + c_{02}\rho_2^2 + O_3 \tag{3.5}$$

Rather complicated computations show that for periodic motions corresponding to equilibria $Q_1 = \pm\pi/2$, $P_1 = P_1^*$ the coefficients of function (3.5) obey the following estimates

$$\delta_1 = \mp\varepsilon\sqrt{|ff''|} + O(\varepsilon^2), \quad \delta_2 = 1 \pm \varepsilon\sqrt{P_1^*} + O(\varepsilon^2)$$

$$c_{20} = \pm\varepsilon\frac{3(c^2 + 4P_1^{*2})}{8ff''^2P_1^{*2}} + O(\varepsilon^2), \quad c_{11} = O(\varepsilon), \quad c_{02} = O(\varepsilon^2)$$

The values of the function f and its derivative f' are evaluated at $P_1 = P_1^*$.

The orbital stability condition $c_{20}\delta_2^2 - c_{11}\delta_1\delta_2 + c_{02}\delta_1^2 \neq 0$ reduces, for small ε, to the inequality $c_{20} \neq 0$, which is obviously satisfied if ε is small enough.

4. CONDITIONALLY PERIODIC MOTIONS

We now consider non-linear oscillations other than the periodic motions studied in Sections 2 and 3. We will confine ourselves to the case of exact commensurability 2:1, assuming that the parameter μ in (3.3) is zero.

Approximate system. If $\mu = 0$, the Hamiltonian of the approximate system is

$$H^* = P_2 + \varepsilon(P_2 - 2P_1)\sqrt{P_1}\sin Q_1 \tag{4.1}$$

In this system, $P_2 = \text{const} > 0$, $dQ_2/d\tau = 1 + \varepsilon\sqrt{(P_1)}\sin Q_1$, and the equations describing the variation of the variables Q_1 and P_1 have a Hamiltonian of the form

$$F = \varepsilon f(P_1)\sin Q_1, \tag{4.2}$$

where f is the function defined in (3.4) and $0 < P_1 \leqslant c/2$. The equations corresponding to the function (4.2)

$$\frac{dQ_1}{d\tau} = \varepsilon f'\sin Q_1, \quad \frac{dP_1}{d\tau} = -\varepsilon f\cos Q_1 \tag{4.3}$$

have the integral

$$F = \varepsilon h = \text{const} \tag{4.4}$$

It follows from the properties of the function f and from the inequality $|\sin Q_1| \leqslant 1$ that in this integral $|h| \leqslant \sqrt{(2/27)}c^{3/2}$.

If $h = \pm\sqrt{(2/27)}c^{3/2}$, we have equilibrium positions $Q_1 = \pm\pi/2$, $P_1 = c/6$ of system (4.3). In the full system with Hamiltonian (3.3) they correspond to the families of orbitally stable long-period motions considered in Section 3.

When $h = 0$ there are two possible types of motion. In motion of one type, $P_1 = c/2$, which corresponds to the Lyapunov short-period motions which, as seen in Section 2, are orbitally unstable when $\mu = 0$. The motions of the second type are asymptotic to the Lyapunov motions as $\tau \to \pm\infty$ (Fig. 1a).

If $0 < |h| < \sqrt{(2/27)}c^{3/2}$, the motion is oscillatory in nature. These motions are shown in Fig. 1(b) in the Q_1, P_1 plane as closed orbits around the equilibrium points $(\pm\pi/2, c/6)$, and in Fig. 1(a), in the P_1, $dP_1/d\tau$ plane, as orbits around the point $(c/6, 0)$. Let us determine an explicit expression for the function $P_1(\tau)$ on these orbits.

Using integral (4.4) to eliminate Q_1 from the second equation of system (4.3), we arrive at the equation

$$\left(\frac{dP_1}{d\tau}\right)^2 = \varepsilon^2(f^2 - h^2) \tag{4.5}$$

The P_1 values corresponding to real oscillatory motion are those for which $f^2 > h^2$.

Equation (4.5) can be integrated in terms of elliptic functions. We obtain

$$P_1 = c\{x_1 + (x_2 - x_1)\operatorname{sn}^2[\varepsilon\sqrt{c(x_3 - x_1)}\tau, k]\} \tag{4.6}$$

$$k^2 = (x_2 - x_1)/(x_3 - x_1), \quad 0 < k^2 < 1$$

where x_j ($j = 1, 2, 3$) are the roots of the equation $\Phi(x) = 0$

$$\Phi(x) = x^3 - x^2 + x/4 - z^2/54 \tag{4.7}$$

Here we have introduced the notation $z = \sqrt{(27/2)}c^{-3/2}h$ $(0 < |z| < 1)$. For the quantities x_j, which are functions of z, we have the following expressions

$$x_j = \frac{2}{3}\cos^2\left[\frac{1}{6}\arccos(2z^2 - 1) + \frac{j\pi}{3}\right], \quad j = 1,2,3 \tag{4.8}$$

It can be verified that $0 < x_1 < 1/6 < x_2 < 1/2 < x_3 < 2/3$.

The minimum and maximum values of $P_1(\tau)$ are denoted in Fig. 1(a) by $c_1 = cx_1$ and $c_2 = cx_2$, respectively.

With the function (4.6) known, one can determine Q_1 from integral (4.4). The frequency ω_1 at which the values of Q_1 and P_1 vary is computed from the formula

$$\omega_1 = \omega_* \frac{\pi\sqrt{2(x_3 - x_1)}}{2K(k)} \tag{4.9}$$

where $K(k)$ is the complete elliptic integral of the first kind and $\omega_* = \varepsilon\sqrt{2c}$ is the frequency of small oscillations of the approximate system in the neighbourhood of its equilibrium points $(\pm\pi/2, c/6)$. The quantity ω_1/ω_* is a function of z. Its graph is shown in Fig. 2. As $z \to 1$ (in the neighbourhood of the equilibria) we have

$$\omega_1 = \omega_*\left[1 - \frac{5}{18}(1 - z) + O((1 - z)^{3/2})\right]$$

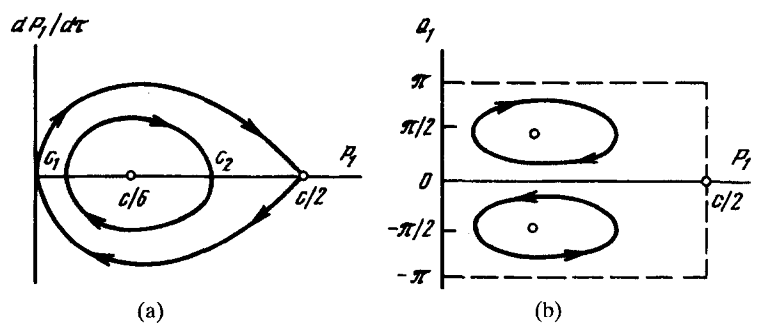

Fig. 1.

and as $z \to 0$ (when the closed orbit in Fig. 1(a) approaches arbitrarily close to the asymptotic orbits) the frequency of the oscillations ω_1 tends to zero, in such a way that

$$\omega_1 \simeq -\pi\omega_* / \ln z$$

Action-angle variables. When investigating the oscillations in the full system it is convenient to introduce action-angle variables [17] for the approximate Hamiltonian (4.1). To fix our ideas, we will consider the oscillations represented in Fig. 1(b) in the Q_1, P_1 plane by the closed orbits around the points ($\pi/2$, $c/6$). Along such orbits, $0 < h < \sqrt{(2/27)}c^{3/2}$, or $0 < z < 1$.

We first introduce action variables I_1 and I_2. Since Q_2 is a cyclic coordinate, it follows that $I_2 = P_2$, while I_1 is obtained from the formula

$$I_1 = \frac{1}{2\pi}\oint P_1 dQ_1 \tag{4.10}$$

where P_1 is defined by the following equality, which follows from (4.4)

$$(I_2 - 2P_1)\sqrt{P_1}\sin Q_1 = h \tag{4.11}$$

and the integral is evaluated along the aforementioned closed orbit in the Q_1, P_1 plane.

Using (4.11) and Eqs (4.3), one can reduce the integral in (4.10) to an integral along the curve in the P_1, $dP_1/d\tau$ plane around the point ($c/6$, 0) in Fig. 1(a). The result is

$$I_1 = \frac{h}{\pi}\int_{c_1}^{c_2} \frac{P_1 f'}{f\sqrt{f^2 - h^2}} dP_1 \tag{4.12}$$

where f is the function in (3.4), with $c = I_2$.

Making the substitution $P_1 = I_2 x$ in (4.12), and noting that

$$h = z\sqrt{2/27}\, I_2^{3/2} \tag{4.13}$$

we obtain the integral

$$I_1 = \frac{\sqrt{6}I_2 z}{36\pi}\int_{x_1}^{x_2} \frac{1-6x}{(1-2x)\sqrt{\Phi(x)}} dx$$

where $\Phi(x)$ is the function (4.7). Integrating, we find that

$$I_1 = I_1^{(s)}\Psi(z) \tag{4.14}$$

where $I_1^{(s)} = -I_2/4$ is the limit value of the action variable I_1 as $z \to 0$, and

$$\Psi(z) = 2\sqrt{6}z\frac{2\Pi(n,k) - 3(1-2x_1)K(k)}{9\pi(1-2x_1)\sqrt{x_3 - x_1}}, \quad n = 2\frac{x_2 - x_1}{1 - 2x_1}, \quad 0 < n < 1 \tag{4.15}$$

where $\Pi(n, k)$ is the complete elliptic integral of the third kind [18] and k is as defined in (4.6). The graph of the function $\Psi(z)$ is shown in Fig. 2.

Equation (4.14) can be solved for z, giving

$$z = \varphi(I_1/I_2) \tag{4.16}$$

In particular, for small I_1 values (i.e., in a small neighbourhood of the equilibrium position) we have

$$z = 1 + 3\sqrt{3}(I_1/I_2) + 15/4(I_1/I_2)^2 + \ldots$$

By (4.13) and (4.16), we can write (4.11) in the form

$$(I_2 - 2P_1)\sqrt{P_1}\sin Q_1 = \sqrt{2/27}\, I_2^{3/2}\varphi(I_1/I_2) \tag{4.17}$$

and Hamiltonian (4.1) of the approximate system may be written as follows:

$$H^* = H^{(0)}(I_2) + \varepsilon H^{(1)}(I_1, I_2) \tag{4.18}$$

$$H^{(0)} = I_2, \quad H^{(1)} = \sqrt{2/27} I_2^{3/2} \varphi(I_1 / I_2)$$

In this situation, the frequency ω_1 of (4.9) satisfies the equality

$$\omega_1 = \varepsilon \partial H^{(1)} / \partial I_1 \tag{4.19}$$

and ω_2 is determined from the formula

$$\omega_2 = 1 + \varepsilon \partial H^{(1)} / \partial I_2 \tag{4.20}$$

The following expression for $\partial H^{(1)}/\partial I_2$ is derived from (4.9), (4.18) and (4.19)

$$\frac{\partial H^{(1)}}{\partial I_2} = \frac{z I_2 K - \pi \sqrt{6(x_3 - x_1)} I_1}{\sqrt{6 I_2} K}$$

We will now show how to obtain a canonical change of variables $Q_i, P_i \to I_i, w_i$ $(i = 1, 2)$ that will yield action-angle variables. The requisite calculations are considerably simplified, since a solution of Eqs (4.3) for the variables Q_1 and P_1 is already known. We will use the following proposition.

Proposition. Let $H(q, p)$ be the Hamiltonian of a system with one degree of freedom. Suppose that in some range of values of the constant of integration $H(q, p) = h$ the motion is periodic in nature and is described by the formulae

$$q = q^*(t + t_0, h), \quad p = p^*(t + t_0, h)$$

Let I be the action variable and let $h(I)$ be the Hamiltonian H expressed in terms of action-angle variables I, w. Then the univalent canonical change of variables $q, p \to I, w$ may be written in the form

$$q = q^*(w/\omega, h), \quad p = p^*(w/\omega, h); \quad h = h(I), \quad \omega = dh/dI$$

The truth of this almost obvious statement may be verified, for example, by checking directly that the Poisson bracket (q^*, p^*) equals unity.

Using this proposition, we obtain from (4.3), (4.4) and (4.6) a change of variables $Q_1, P_1 \to I_1, w_1$ in the form

$$Q_1 = \pi - \arccos \frac{n \operatorname{sn} u \operatorname{cn} u \operatorname{dn} u}{(1 - n \operatorname{sn}^2 u)\sqrt{s + k^2 \operatorname{sn}^2 u}} \tag{4.21}$$

$$P_1 = I_2[x_2 + (x_2 - x_1)\operatorname{sn}^2 u]; \quad u = K w_1 / \pi, \quad s = x_1 / (x_3 - x_1)$$

where k^2 is given by (4.6) and n is as defined in (4.15); the quantities x_j are calculated by formulae (4.8) and (4.16).

The change of variables (4.21) could also have been obtained using a generating function of the form

$$S_1(I_1, I_2, Q_1) = \int_{\pi/2}^{Q_1} P_1 dQ_1$$

The function $P_1 = P_1(I_1, I_2, Q_1)$ is determined from (4.17).

The change of variables (4.21), which is canonical with respect to I_1 and w_1, transforms the left-hand side of equality (4.17) into its right-hand side $H^{(1)}$. The transformation involves I_2 as a parameter. To obtain a transformation $Q_i, P_i \to I_i, w_i$ $(i = 1, 2)$ which is also canonical in the variables I_2 and w_2 and reduces Hamiltonian (4.1) to the form (4.18), we introduce a generating function

$$S(I_1, I_2, Q_1, Q_2) = I_2 Q_2 + S_1$$

Then, as might be expected

$$w_2 = Q_2 + \int_{\pi/2}^{Q_1} \frac{\partial P_1}{\partial I_2} dQ_1$$

Hence, reasoning from relations (4.3), (4.17) and (4.21), one can find $Q_2 = w_2 + g(I_1, I_2, w_1)$. The explicit form of the function g will not be written out here as it will not be needed in what follows.

Preservation of conditionally periodic motions in the full system. If $0 < h < \sqrt{(2/27)}c^{3/2}$, the motion in the approximate system is oscillatory. The frequencies of the oscillations are determined by formulae (4.9) and (4.20). If the ratio $\omega_1:\omega_2$ for given initial data is not a rational number, the motion will be conditionally periodic.

We will now consider the full system whose Hamiltonian is given by (3.3), but putting $\mu = 0$. In terms of the variables I_1 and w_1 ($i = 1, 2$), this Hamiltonian will be

$$H = H^{(0)}(I_2) + \varepsilon H^{(1)}(I_1, I_2) + \varepsilon^2 H^{(2)}(I_1, I_2, w_1, w_2; \varepsilon) \tag{4.22}$$

where $H^{(0)}$ and $H^{(1)}$ are functions as in (4.18) and $H^{(2)}$ is defined in the oscillation domain $0 < h < \sqrt{(2/27)}c^{3/2}$ as a 2π-periodic function of w_1 and w_2, analytic in all its arguments.

When $\varepsilon = 0$, Hamiltonian (4.22) depends on only one of the action variables I_2. Consequently, this is a case of proper degeneracy [14]. At the same time

$$\frac{\partial H^{(0)}}{\partial I_2} \neq 0, \quad \frac{\partial H^{(1)}}{\partial I_1} \neq 0, \quad \frac{\partial^2 H^{(1)}}{\partial I_1^2} \neq 0 \tag{4.23}$$

The truth of the first two relations follows at once from (4.9), (4.18) and (4.19). Let us verify the third. Computations show that

$$\frac{\partial^2 H^{(1)}}{\partial I_1^2} = \frac{5G(z)}{\sqrt{6I_2}}$$

$$G(z) = \pi^2 z \frac{K(x_3 - x_2)[2(x_2 - x_1)(3x_2 - 1) - 1] + E(x_3 - x_1)}{60K^3(x_3 - x_1)(x_3 - x_2)^2(x_2 - x_1)^2}$$

where $E = E(k^2)$ is the complete elliptic integral of the second kind.

If $0 < z < 1$, $G(z)$ is positive and monotone decreasing; moreover, if $z \to 1$, then $G(z) \to 1$, and if $z \to 0$, then

$$G \simeq -\frac{18\pi^2}{5z\ln^3 z}$$

A graph of $G(z)$ is shown in Fig. 2.

Thus all three relations (4.23) are true.

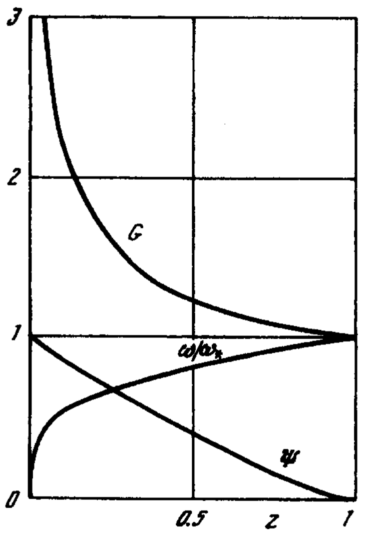

Fig. 2.

Consequently, by previously known results [14, 19], when $0 < h < \sqrt{(2/27)}c^{3/2}$ the motion of the full system with Hamiltonian (3.3) is conditionally periodic with frequencies ω_1 and ω_2, for most initial data. Only a part $Q(\exp(-a_1\varepsilon^{-1}))$, where $a_1 = \text{const} > 0$, of the phase space is not filled by conditionally periodic orbits. Under these conditions, for all initial data the quantities I_i $(i = 1, 2)$ are for all τ close to their initial values

$$|I_i(\tau) - I_i(0)| < a_2\varepsilon \ (a_2 = \text{const})$$

5. OSCILLATIONS OF AN ELASTIC PENDULUM

Following the approach of Vitt and Gorelik [3], let us consider the problem of non-linear resonant oscillations of an elastic pendulum. The pendulum is a point mass, attached at one end of a weightless spring whose other end is fastened to a fixed point O. Motion takes place in a fixed vertical plane in a uniform gravitational field. Suppose m is the mass of the point, g is the acceleration due to gravity, l_0 is the length of the undeformed spring, and k is its stiffness. Let ρ and θ denote the polar coordinates of the point, with the angle θ measured from an axis passing through the point O at which the spring is attached and pointing vertically downwards.

The pendulum may be in a state of equilibrium on that axis. Then $\theta = 0$ and $\rho = l$, where $l_0 = l(1-\chi^2)$. Here $\chi = \sqrt{(mg/(kl))}$ is the ratio of the frequency $\sqrt{(g/l)}$ of small horizontal oscillations of the point in the vicinity of equilibrium to the frequency $\sqrt{(k/m)}$ of its small oscillations along the vertical.

Let us investigate the pendulum in the neighbourhood of this equilibrium position. As generalized coordinates we take the dimensionless quantities ξ and η defined by

$$\xi = \rho/l - 1, \quad \eta = \sqrt{\chi}\theta$$

Further, introducing dimensionless momenta

$$p_\xi = \sqrt{m/k}\dot{\xi}, \quad p_\eta = \sqrt{l/g}(1+\xi)^2\dot{\eta}$$

and transforming to dimensionless time $\tau = \sqrt{(g/l)}t$, we obtain the Hamiltonian

$$H = \frac{1}{2\chi}\left[p_\xi^2 + \chi\frac{p_\eta^2}{(1+\xi)^2}\right] - \chi(1+\xi)\cos\frac{\eta}{\sqrt{\chi}} + \frac{1}{2\chi}(\chi^2+\xi)^2 \tag{5.1}$$

The solution of the equations of motion corresponding to equilibrium of the point on the vertical is $\xi = \eta = p_\xi = p_\eta = 0$. The expansion of Hamiltonian (5.1) in the neighbourhood of this solution is

$$H = \frac{1}{2\chi}(\xi^2 + p_\xi^2) + \frac{1}{2}(\eta^2 + p_\eta^2) + \frac{1}{2}\xi(\eta^2 - 2p_\eta^2) + \ldots \tag{5.2}$$

The unimportant constant term in this expansion is omitted.

In keeping with our assumption that we are investigating motions of the pendulum near its equilibrium position, we introduce a small parameter ε through the change of variables

$$\xi = \varepsilon\tilde{\xi}, \quad \eta = \varepsilon\tilde{\eta}, \quad p_\xi = \varepsilon\tilde{p}_\xi, \quad p_\eta = \varepsilon\tilde{p}_\eta, \quad 0 < \varepsilon \ll 1$$

This change of variables is canonical with valence ε^{-2}. The Hamiltonian corresponding to the equations of motion in the new variables is

$$H = \frac{1}{2\chi}(\tilde{\xi}^2 + \tilde{p}_\xi^2) + \frac{1}{2}(\tilde{\eta}^2 + \tilde{p}_\eta^2) + \frac{\varepsilon}{2}\tilde{\xi}(\tilde{\eta}^2 - 2\tilde{p}_\eta^2) + O(\varepsilon^2) \tag{5.3}$$

Let us assume that twice the frequency of small horizontal oscillations of the pendulum is close to the frequency of its small vertical oscillations. Introducing yet another small parameter, the "frequency difference" μ, we put

$$\sqrt{k/m} = \sqrt{g/l}(2+\mu), \quad (0 \leqslant |\mu| \ll 1)$$

We will assume that μ is of order of magnitude at least 1 relative to ε.

We apply one more canonical transformation (with valence 9/8)

$$
\begin{aligned}
\tilde{\xi} &= -\frac{2\sqrt{2}}{3}x_1 - \frac{\varepsilon}{18}(x_2^2 - 5y_2^2) + O(\varepsilon^2) \\
\tilde{\eta} &= -\frac{2\sqrt{2}}{3}x_2 + \varepsilon\frac{5}{9}y_1y_2 + O(\varepsilon^2) \\
\tilde{p}_\xi &= -\frac{2\sqrt{2}}{3}y_1 + O(\varepsilon^2) \\
\tilde{p}_\eta &= -\frac{2\sqrt{2}}{3}y_2 + \frac{\varepsilon}{9}x_2y_1 + O(\varepsilon^2)
\end{aligned}
\tag{5.4}
$$

In the new variables x_i, y_i ($i = 1, 2$), the motion is described by canonical equations with Hamiltonian (1.2). The oscillations of the pendulum may now be analysed on the basis of the general conclusions derived above in Sections 2–4 for oscillations of an autonomous Hamiltonian system with 2:1 resonance.

Short-period motions. In our problem concerning the oscillations of an elastic pendulum where the elastic force obeys a linear law, the Lyapunov short-period motions are harmonic oscillations of the point mass along a vertical axis passing through the point of attachment O of the spring. For these oscillations we have

$$
\theta = 0, \quad \rho = l\left\{1 - \varepsilon\frac{2\sqrt{2c}}{3}\sin\left[\sqrt{\frac{k}{m}}(t + t_0)\right]\right\}
\tag{5.5}
$$

where t_0 and c are arbitrary constants ($c > 0$). If inequality (2.5) is true, then, for sufficiently small ε, these oscillations will be orbitally stable. But if the reverse inequality to (2.5) is true, we have orbital instability. This is in agreement with well-known results obtained by analytical investigation of the stability of low-amplitude vertical oscillations (see, e.g. [20]). The orbital stability of vertical oscillations of arbitrary amplitude was studied in [21] by numerical methods.

Long-period motions. According to Section 3, long-period motions emanate from equilibrium positions of the system for which $Q_1 = \pm\,\pi/2$, and the corresponding P_1 values are the roots of the equation $\mu \pm f' = 0$. Denoting these values by $P_1^\pm$ (the upper sign for $Q_1 = \pi/2$ and the lower sign for $Q_1 = -\pi/2$), we find that

$$
P_1^\pm = [\sqrt{(\mu/\varepsilon)^2 + 6c} \pm \mu/\varepsilon]^2/36
$$

Denote the long-period motions corresponding to equilibria $Q_1 = \pm\,\pi/2, P_1 = P_1^\pm$ by $\Pi^\pm$. In accordance with the canonical transformations of this section and Section 3, these periodic motions of the point mass are described in terms of the original polar coordinates ρ, θ by the formulae

$$
\begin{aligned}
\rho &= l\left(1 \mp \frac{4}{3}\varepsilon\sqrt{P_1^\pm}\cos 2Q_2 + O(\varepsilon^2)\right) \\
\theta &= -\frac{4}{3}\varepsilon\sqrt{2(c - 2P_1^\pm)}\sin Q_2 + O(\varepsilon^2) \\
Q_2 &= \left(1 \pm \varepsilon\sqrt{P_1^\pm} + O(\varepsilon^2)\right)\sqrt{g/l}(t + t_0)
\end{aligned}
\tag{5.6}
$$

If we ignore $O(\varepsilon^2)$ quantities in (5.6), the orbits of the point mass in the motions $\Pi^\pm$ will be parabolas, described twice per period. In Π^+ (Π^-) the branches of the parabola point downwards (upwards). The existence of the motions $\Pi^\pm$ has been pointed out in [3] (see also [22]).

To describe the bifurcations of the periodic motions, as discussed in Section 3, we consider three cases.

1. $\mu < -\varepsilon\sqrt{(2c)}$. In this case the Lyapunov short-period motions (5.5) are orbitally stable and one more family Π^+ of orbitally stable long-period motions exists. In this case, $0 < P_1^+ < c/18$.

2. $|\mu| < \varepsilon\sqrt{(2c)}$. As the boundary $\mu = -\varepsilon\sqrt{(2c)}$ of the stability domain of Lyapunov motions is crossed into their instability domain $|\mu| < \varepsilon\sqrt{(2c)}$, the family Π^- of orbitally stable long-period motions is

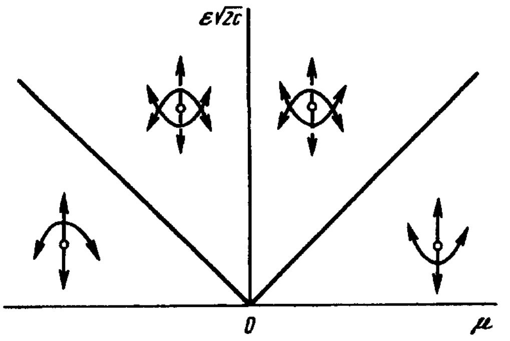

Fig. 3.

generated. Thus, there are three families of periodic motions in the domain $|\mu| < \varepsilon\sqrt{(2c)}$: an orbitally unstable family of Lyapunov short-period motions and two orbitally stable families $\Pi^{\pm}$ of long-period motions. The following relations hold

$$c/18 < P_1^+ < c/6 < P_1^- < c/2 \quad \text{for} \quad -\varepsilon\sqrt{2c} < \mu < 0$$

$$P_1^+ = P_1^- = c/6 \quad \text{for} \quad \mu = 0$$

$$c/18 < P_1^- < c/6 < P_1^+ < c/2 \quad \text{for} \quad 0 < \mu < \varepsilon\sqrt{2c}$$

3. $\mu > \varepsilon\sqrt{(2c)}$. As the boundary $\mu = \varepsilon\sqrt{(2c)}$ of the instability domain of Lyapunov motions is crossed into their stability domain $\mu > \varepsilon\sqrt{(2c)}$, the family Π^+ of long-period motions disappears. Thus, there are two families of periodic motions in the domain $\mu > \varepsilon\sqrt{(2c)}$: Lyapunov motions and Π^- motions, with $0 < P_1^- < c/18$.

The bifurcation portrait just described is outlined in Fig. 3 in the μ, $\varepsilon\sqrt{(2c)}$ plane. The boundaries of the instability domain of the Lyapunov short-period motions are represented in Fig. 3 by the bisectors of the first and second coordinate angles. The $\Pi^{\pm}$ motions are represented by the parabolas, while the Lyapunov motions are represented by vertical segments, the solid lines corresponding to orbitally stable motions and the dashed lines to unstable ones.

Energy "transfer" between vertical and horizontal oscillations. One of the most important non-linear effects in the motion of an autonomous Hamiltonian system in the vicinity of a stable equilibrium position with 2:1 resonance is the possibility of periodic energy transfer between its degrees of freedom. The case of a system with two degrees of freedom has been investigated in detail, using the elastic pendulum as an example [3]. Asymptotic methods, such as the method of averaging, were used there.

It is interesting to clarify the situation with regard to energy transfer in the full and not only the approximate system. It follows from the results of the last part of Section 4 that, in a sufficiently small neighbourhood of an equilibrium position of an elastic pendulum, the phenomenon of periodic energy transfer between its vertical and horizontal oscillations exists over an infinite interval of time, for the majority of initial data. The measure of the set of initial data in phase space for which energy transfer may not occur is exponentially small.

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